Muon Transverse Ionization Cooling:
Stochastic Approach

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Abstract

The muon transverse ionization cooling is modeled as a Brownian motion of the muon beam as it traverses a Li or Be rod. A Langevin like equation is written for the free particle case (no external transverse magnetic field) and for the most realistic harmonically bound beam in the presence of a focusing magnetic field.

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I. INTRODUCTION

The possibility of a $\mu^+ - \mu^-$ collider to explore the Higgs energy range and supersymmetry has begun to be vigorously examined. One of the crucial issues to achieve the required luminosity ($\mathcal{L} \approx 10^{34}\, cm^{-2} s{-1}$) is the need to compress the phase space by means of muon cooling. A technique that has shown to be very promising is ionization cooling. The introduction of the concept and the physics was first discussed by Skrinsky [1]; for a clear and comprehensive treatment we refer the reader to Neuffer's article [2].

Neuffer has suggested the following differential equation, or difference equation for long sections,

$$\frac{d\epsilon_\perp^N}{dz} = -|\frac{dE_\mu}{dz}|\frac{\epsilon_\perp^N}{E_\mu} + \frac{1}{2}\beta_\perp \gamma\beta \frac{d < \theta^2 >}{dz} \tag{1}$$

where $\epsilon_\perp^N$ is the muons normalized transverse emittance, E_μ is the muon total energy, $\beta_\perp$ is the beta function and $< \theta^2 >$ is the square of the rms divergence of the beam due to multiple scattering.

The original derivation in ref. [2] assumed a cooling system consisting of small alternating absorber and reaccelerator sections; subsequently, Palmer [3] and Fernow [4] have argued that Eq.1 is valid for a single absorber. Their argument is quite straightforward; from the definition of emittance $\epsilon_\perp^N = \gamma\beta\sqrt{< r^2 >< \theta^2 >}$ compute the derivative with respect to z

$$\frac{d\epsilon_\perp^N}{dz} = \frac{d(\gamma\beta)}{dz}\frac{\epsilon_\perp^N(z)}{\gamma\beta} + \frac{1}{2}\frac{\epsilon_\perp^N(z)}{< r^2 >}\frac{d < r^2 >}{dz} + \frac{1}{2}\frac{\epsilon_\perp^N(z)}{< \theta^2 >}\frac{d < \theta^2 >}{dz} \tag{2}$$

Recalling $\frac{d(\gamma\beta)}{dz} = \frac{1}{\beta mc^2}\frac{dE_\mu}{dz}$, $\epsilon_\perp^N \beta_\perp = \gamma\beta < r^2 >$ and $\frac{\epsilon_\perp^N}{\beta_\perp} = \gamma\beta < \theta^2 >$ we obtain a slightly modified version of Eq. 1,

$$\frac{d\epsilon_\perp^N}{dz} = -|\frac{dE_\mu}{dz}|\frac{\epsilon_\perp^N}{\beta^2 E_\mu} + \frac{1}{2}\gamma\beta\beta_\perp \frac{d < \theta^2 >}{dz} + \frac{1}{2}\frac{\gamma\beta}{\beta_\perp}\frac{d < r^2 >}{dz} \tag{3}$$

The first term reflects the energy loss (*cooling*) and the last two terms are produced by multiple scattering (*heating*).

Assuming a long cooling rod (Li or Be) the gaussian approximation is quite adequate, then

$$< y^2 >= \frac{1}{3}\theta_c^2 z^3 \qquad < \theta^2 >= \theta_c^2 z \tag{4}$$

where the projected angle $\theta_o \equiv \theta_c\sqrt{z} = \frac{13.6[MeV]}{\beta cp}\sqrt{\frac{z}{L_R}}$; in this expression we have neglected logarithmic correction terms [5]. We could also use the more accurate formula [6]

$$\theta_o = \frac{\chi_c}{\sqrt{1+F^2}}\sqrt{\frac{1+\nu}{\nu}\log(1+\nu) - 1} \tag{5}$$

with the characteristic angle $\chi_c = \frac{\sqrt{0.157[MeV]z}}{\beta p}\sqrt{\frac{Z(Z+1)}{A}}$; the phenomenological parameters $F = 0.98$ and $\nu = \frac{\Omega_o}{2(1-F)}$ are chosen to fit the experimental data (Ω_o represent the mean number of scatters in the medium) and L_R is the radiation length.

It has been argued [3]that smaller $\beta_\perp$ produces smaller transverse heating; however, as we see from Eq.3, the third term increases with small $\beta_\perp$. Therefore, if Eq.3 describes the physical situation, the minimum heating is obtained when

$$\beta_\perp(0) = \sqrt{\frac{\frac{d<y^2>}{dz}}{\frac{d<\theta^2>}{dz}}} = L_{rod} \tag{6}$$

In other words the beta function of the incident beam on the cooling rod *must* match the length of the rod; furthermore the minimum achievable emittance is, $\epsilon_\perp^N|_{min} = \frac{(13.6)^2}{2\beta mc^2|\frac{dE_\mu}{dz}|}\frac{L_{rod}}{L_R}$. Neuffer [2] has derived a similar expression with L_{rod} replaced by $\beta_\perp$,

$$\epsilon_\perp^N|_{min} = \frac{(13.6[MeV])^2}{2\beta mc^2|\frac{dE_\mu}{dz}|}\frac{\beta_\perp(0)}{L_R} \tag{7}$$

Subsequently, Palmer [3] has conjectured that the third term in Eq.3 need not be included because the multiple scattering medium (Li or Be rod) is immersed in a uniform transverse magnetic field, preventing the beam from spreading laterally. This raises the question, how is the particle distribution changed in position and angle due to the external magnetic field? We will examine this question in the next sections.

We should also point out that the treatment considered here is also of interest for a number of other problems in accelerator physics. These include scattering of particles by residual gas in a synchrotron [7], scattering in the Inverse Čerenkov accelerators [8], plasma beat-wave accelerators [9] and plasma lenses for future linear colliders [10] and the dynamics of space-charge dominated beams [11].

II. PARTICLE DISTRIBUTION WITHOUT EXTERNAL MAGNETIC FIELD

This problem has been analyzed in detail by Rossi [12] following the Fokker-Planck equation approach. Let $\mathcal{W}(y,\theta,z;y_o,\theta_o)\,dy\,d\theta$ represent the number of particles in the phase space element $(y, y+dy; \theta, \theta+d\theta)$ after traversing a medium of thickness z and initial coordinates $y(0)=y_0$, $\theta(0)=\theta_o$. It satisfies the equation

$$\frac{\partial \mathcal{W}}{\partial z} = -(\theta-\Theta_o)\frac{\partial \mathcal{W}}{\partial y} + \frac{\theta_c^2}{2}\frac{\partial^2 \mathcal{W}}{\partial \theta^2} \tag{8}$$

with boundary conditions $\mathcal{W}(y,\theta,z;y_o,\theta_o)|_{z=0} = \delta(y-y_o)\delta(\theta-\theta_o)$ and solution

$$\mathcal{W}(y,\theta,z:y_o,\theta_o) = \frac{\sqrt{3}}{\pi z^2\theta_c^2}\exp\left[-\frac{2}{\theta_c^2}\left(\frac{(\theta-\Theta_o)^2}{z} - \frac{3(y-Y_o)(\theta-\Theta_o)}{z^2} + \frac{3(y-Y_o)^2}{z^3}\right)\right] \tag{9}$$

which can be verified by direct substitution; $\Theta_o = \theta_o$ and $Y_o = y_o + \theta_o z$. This result allows us to compute the emittance of the beam after traversing the cooling rod. After tedious gaussian integration we obtain

$$\begin{aligned}
&< y >= Y_o \\
&< \theta >= \theta_o \\
&< y^2 >= Y_o^2 + \tfrac{\theta_c^2 z^3}{3} \\
&< \theta^2 >= \Theta_o^2 + \theta_c^2 z \\
&< y\theta >= \Theta_o Y_o + \tfrac{\theta_c^2 z^2}{2}
\end{aligned} \tag{10}$$

Averaging over the initial coordinates assuming gaussian distributions, we obtain

$$\begin{aligned}
&\ll y \gg = \qquad \ll \theta \gg = 0 \\
&\ll y^2 \gg = \sigma_{yo}^2 + \sigma_{\theta o}^2 z^2 + 2z < y_o\theta_o > + \tfrac{\theta_c^2 z^3}{3} \\
&\ll \theta^2 \gg = \sigma_{\theta o}^2 + \theta_c^2 z \\
&\ll y\theta \gg = \sigma_{\theta o}^2 z + < y_o\theta_o > + \tfrac{\theta_c^2 z^2}{2}
\end{aligned} \tag{11}$$

and the total emittance in the absence of a focusing field is

$$\epsilon_\perp(z) = \sqrt{\epsilon_\perp^2(0) + \frac{\theta_c^4 z^4}{12} + \sigma_{\theta o}^2\theta_c^2\frac{z^3}{3} + \theta_c^2 < y_o\theta_o > z^2 + \sigma_{yo}^2\theta_c^2 z} \tag{12}$$

The terms proportional to θ_c are the contributions due to multiple scattering.

III. PARTICLE DISTRIBUTION WITH EXTERNAL MAGNETIC FIELD

In general a particle in a transverse magnetic field satisfies the equation of motion $\frac{d^2y}{dz^2} + K(z)y = 0$ where $K(z) = \frac{eB}{mc\gamma\beta a} = \omega^2$, B is the azimuthal magnetic field and a is the radius of the channel (radius of the rod); $K(z)$ is a function of z because of the energy loss. For simplicity of the arguments that follow, we neglect the energy change as the beam traverses the rod.

This is not the complete picture because the particle suffers random accelerations due to scattering (i.e. stochastic changes in angle $\frac{dy}{dz} = \theta$). A correct equation of motion is

$$\frac{dy}{dz} = \theta \quad , \quad \frac{d\theta}{dz} + K(z)y = A(z) \tag{13}$$

where we denote with $A(z)$ the random acceleration due to Coulomb scattering which excites betatron oscillations in the beam. This equation is formally a Langevin equation of a particle in an external field K(z)y (harmonic oscillator) where the frequency is a function of the *time* variable z. The main assumptions regarding the stochastic variable $A(z)$, more precisely $\int_z^{z+dz} dz' A(z')$, is that it is independent of y, that it varies extremely rapidly compared to the variations of the coordinates y and θ, and that it is Gaussian-distributed with a variance θ_c^2.

Therefore, we cast the muon cooling problem as an stochastic one; we have to find the particle distribution $\mathcal{W}(y, \theta, z; y_o \theta_o)$, and as before, from that function we can calculate the emittance.

The method to determine the distribution function uses standard techniques for solving ordinary differential equations [13]. We propose as a solution $y(z) = a_1(z)\exp(\lambda z) + a_2(z)\exp(-\lambda z)$. After some manipulations we obtain the following relations,

$$\begin{aligned} y(z) - y_o \cos\omega z - \theta_o z \frac{\sin\omega z}{\omega z} &= \int_{-\infty}^{+\infty} dz' A(z')\Psi(z') \\ \theta(z) - \theta_o \cos\omega z + y_o \omega \sin\omega z &= \int_{-\infty}^{+\infty} dz' A(z')\Phi(z') \end{aligned} \tag{14}$$

with $\Psi(z, z') = \frac{\sin\omega(z-z')}{\omega}$ and $\Phi(z, z') = \cos\omega(z - z')$.

A fundamental Lemma in the theory of stochastic differential equations [13], [14] states that for $y(z) - Y_o(z) = \int_0^z dz'\Psi(z')A(z')$ and $\theta - \Theta_o(z) = \int_0^z dz'\Phi(z')A(z')$, the most general solution is a particle distribution of the form

$$\mathcal{W}(y,\theta,z,\omega;y_o,\omega_o) = \frac{1}{2\pi\sqrt{\{FG-H^2\}}} \exp\left[-\frac{1}{2\{FG-H^2\}}\left(G(y-Y_o)^2 - 2H(y-Y_o)(\theta-\Theta_o) + F(\theta-\Theta_o)^2\right)\right] \quad (15)$$

where the parameters F,G H are function of the external focusing field;

$$\begin{aligned} F &= \theta_c^2 \tfrac{z}{2\omega^2}\left(1 - \tfrac{\sin 2\omega z}{2\omega z}\right) \\ G &= \theta_c^2 \tfrac{z}{2}\left(1 + \tfrac{\sin 2\omega z}{2\omega z}\right) \\ H &= \theta_c^2 \tfrac{1}{2\omega^2}\tfrac{1}{2}(1 - \cos 2\omega z) \\ FG - H^2 &= \tfrac{\theta_c^4}{\omega^4} z^2 \left[1 - \left(\tfrac{\sin\omega z}{\omega z}\right)^2\right] \end{aligned} \quad (16)$$

It can be shown that Eq.15 reproduces Eq.9 in the limit $\omega \rightarrow 0$.The probability density $\mathcal{W}(y,\theta,z,\omega;y_o,\theta_o)$ satisfies a parabolic partial differential equation, the Fokker-Planck equation

$$\frac{\partial \mathcal{W}}{\partial z} = -(\theta - \Theta_0)\frac{\partial \mathcal{W}}{\partial y} + \omega^2(y - Y_o)\frac{\partial \mathcal{W}}{\partial \theta} + \frac{1}{2}\theta_c^2\frac{\partial^2 \mathcal{W}}{\partial \theta^2} \quad (17)$$

As in the previous section we are interested in calculating the second moments of the distribution and from those the emittance; the algebra is involved but trivial and we get,

$$\begin{aligned} <y> &= y_o \cos\omega z + \theta_o z \tfrac{\sin\omega z}{\omega z} \\ <\theta> &= \theta_o \cos\omega z - y_o\omega \sin\omega z \\ <y^2> &= \left(y_o \cos\omega z + \theta_o z \tfrac{\sin\omega z}{\omega z}\right)^2 + z\tfrac{\theta_c^2}{2\omega^2}\left(1 - \tfrac{\sin 2\omega z}{2\omega z}\right) \\ <\theta^2> &= (\theta_o \cos\omega z - y_o\omega\sin\omega z)^2 + z\tfrac{\theta_c^2}{2}\left(1 + \tfrac{\sin 2\omega z}{2\omega z}\right) \\ <y\theta> &= y_o\theta_o \cos 2\omega z - \tfrac{y_o^2\omega}{2}\sin 2\omega z + \theta_o^2 z\cos\omega z\tfrac{\sin\omega z}{\omega z} + \tfrac{\theta_c^2 z^2}{2}\left(\tfrac{\sin\omega z}{\omega z}\right)^2 \end{aligned} \quad (18)$$

If we now assume an uncorrelated ensemble of particles with independent gaussian distributions of initial conditions y_o and θ_o and average over both variables, we obtain

$$
\begin{aligned}
\ll y \gg &= \qquad\qquad \ll \theta \gg = 0 \\
\ll y^2 \gg &= \sigma_{yo}^2(\cos\omega z)^2 + \sigma_{\theta o}^2 z^2 \left(\frac{\sin\omega z}{\omega z}\right)^2 + z\frac{\theta_c^2}{2\omega^2}\left(1 - \frac{\sin 2\omega z}{2\omega z}\right) \\
\ll \theta^2 \gg &= \sigma_{\theta o}^2(\cos\omega z)^2 + \sigma_{yo}^2\omega^2(\sin\omega z)^2 + z\frac{\theta_c^2}{2}\left(1 + \frac{\sin 2\omega z}{2\omega z}\right) \\
\ll y\theta \gg &= \sigma_{\theta o}^2 z\frac{\sin 2\omega z}{2\omega z} - \sigma_{yo}^2\frac{1}{2}\omega\sin 2\omega z + \frac{\theta_c^2 z^2}{2}\left(\frac{\sin\omega z}{\omega z}\right)^2
\end{aligned} \tag{19}
$$

An important observation is that for very high magnetic field ($\omega \rightarrow \infty$) the rms beam size remains approximately constant, confirming the conjecture of Palmer [3] mentioned earlier. If the beam is focused to a waist at the entrance to the rod, the emittance at any distance z inside the rod is [15]

$$
\epsilon_\perp^2(z) = \epsilon_\perp^2(0) + \sigma_{yo}^2\theta_c^2 z + \frac{\theta_c^4}{4\omega^2}z^2 + \frac{\theta_c^4}{8\omega^4}\left[1 - \cos(2\omega z)\right] \tag{20}
$$

This also leads to Eq.7 for the minimum emittance, provided that $\sigma_{yo}^2 \gg \frac{\theta_c^2}{2\omega^2}L_{rod}$ and $\sigma_{yo}^2 \gg \frac{\theta_c^2}{4\omega^3}$, which will usually be the case for strong focusing ($\omega \gg 0$).

IV. CONCLUSIONS

Using an analogy with a random dynamical process modelled with a Langevin equation, we have incorporated the stochastic nature of both, position and angle variables into the problem of a muon traversing a Li or Be rod immersed in a uniform azimuthal magnetic field. The pseudo-Brownian motion of the particles in the medium represents heating of the beam. Emittance increase due to Coulomb multiple scattering (the less likely single and plural scattering events are neglected) compensates the cooling, emittance decrease, introduced by the energy loss $\frac{dE_\mu}{dz}$.

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